3. Maximize $3xy - x^3$ subject to the constraints

$$2x - y = -5,$$
$$5x + 2y \geqslant 37,$$
$$x \geqslant 0,\ y \geqslant 0.$$

3

NDCQ: $(2\ \ -1) \neq \vec{0}$, $\begin{pmatrix} 2 & -1 \\ 5 & 2 \end{pmatrix} \neq \vec{0}$, $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \neq \vec{0}$, $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \neq \vec{0}$.

Thus, all other combinations of equality constraint gradient and inequality constrains gradients are also non $\vec{0}$. Thus, NDCQ is fulfilled. ($\forall x, y$).

$$L(x,y;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = 3xy - x^3 + \lambda_1(-5-2x+y) + \lambda_2(5x+2y-37) + \lambda_3 x + \lambda_4 y$$

Mixed constraints.

2

FOC:
$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \cancel{\lambda_2(-5-2x+y)} \quad \lambda_2(5x+2y-37) = 0 \\ \lambda_3 x = 0 \\ \lambda_4 y = 0 \\ \lambda_2,\ \lambda_3,\ \lambda_4 \geq 0 \\ 5x + 2y \geq 37 \\ x \geq 0,\ y \geq 0 \\ \frac{\partial L}{\partial \lambda_1} = 0 \end{cases}$$

$$\begin{cases} 3y - 3x^2 - 2\lambda_1 + 5\lambda_2 + \lambda_3 = 0 \\ 3x + \lambda_1 + 2\lambda_2 + \lambda_4 = 0 \\ \lambda_2(5x+2y-37) = 0 \\ \lambda_3 x = 0 \\ \lambda_4 y = 0 \\ 2x - y = -5 \end{cases} \qquad \begin{cases} 6x + 15 - 3x^2 - 2\lambda_1 + 5\lambda_2 + \lambda_3 = 0 \\ 3x + \lambda_1 + 2\lambda_2 + \lambda_4 = 0 \\ \lambda_2(9x - 27) = 0 \\ \lambda_3 x = 0 \\ \lambda_4(2x+5) = 0 \\ y = 2x + 5 \end{cases}$$

1) $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = 0$, $y = 2x + 5$

$$\begin{cases} 6x + 15 - 3x^2 - 2\lambda_1 = 0 \\ 3x + \lambda_1 = 0 \end{cases}$$

$\lambda_1 = -3x$

$6x + 15 - 3x^2 + 6x = 0$

$x^2 - 4x - 5 = 0$

$x_* = -1$ | $x = 5$
$y = 3$ | $y = 15$
$\lambda_1 = 3$ | $\lambda_1 = -15$

$5 \cdot (-1) + 2 \cdot 3 < 37$ $\quad\quad$ $5 \cdot 5 + 2 \cdot 15 > 37$

$\Downarrow$ $\quad\quad$ $\Downarrow$

$\emptyset$ $\quad\quad$ $M_1 = (5, 15; -15, 0, 0, 0)$

2) $5x + 2y - 37 = 5x + 4x + 10 - 37 = 9x - 27$
$x = 3, \; y = 2 \cdot 3 + 5 = 11; \; \lambda_3, \lambda_4 = 0$

$$\begin{cases} 33 - 27 - 2\lambda_1 + 5\lambda_2 = 0 \\ 9 + \lambda_1 + 2\lambda_2 = 0 \end{cases}$$

$6 + 18 + 4\lambda_2 + 5\lambda_2 = 0, \quad \lambda_2 = -\frac{8}{3} < 0 \Rightarrow \emptyset$

$\lambda_1 = -2\lambda_2 - 9 = \frac{16 - 27}{3} = -\frac{11}{3}$

3) $\lambda_2 = 0, \quad x = 3, \quad \lambda_3 = 0, \quad \lambda_4 = 0$
$y = 11$

$$\begin{cases} 33 - 27 - 2\lambda_1 = 0 \\ 9 + \lambda_1 = 0 \end{cases} \Rightarrow \emptyset$$

4) $\lambda_2 = 0, \quad x = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0$

$$\begin{cases} 15 - 2\lambda_1 = 0 \\ \lambda_1 = 0 \end{cases} \quad \emptyset$$

5) $\lambda_2 = 0, \quad \lambda_3 = 0, \quad x = -\frac{5}{2}$

$$\begin{cases} -15 + 15 - 3 \cdot \frac{25}{4} - 2\lambda_1 = 0 \\ -\frac{15}{2} + \lambda_1 + \lambda_4 = 0 \end{cases}$$

$\lambda_1 = -\frac{75}{8}, \quad \lambda_4 = \frac{135}{8}$

$y = -5 + 5 = 0$

$5 \cdot \left(-\frac{5}{2}\right) + 2 \cdot 0 < 0 \Rightarrow \emptyset$

6) $\lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0, \quad x = -\frac{5}{2}$

$$\begin{cases} -15 + 15 - 3 \cdot \frac{25}{4} - 2\lambda_1 = 0 \\ -\frac{15}{2} + \lambda_1 = 0 \end{cases} \Rightarrow \emptyset$$

7) $\lambda_2 = 0$, $x = 0$, $\lambda_4 = 0$

$$\begin{cases} 15 - 2\lambda_1 + \lambda_3 = 0 \\ \lambda_1 = 0 \end{cases}$$

$\lambda_3 = 15$, $y = 5$

$5 \cdot 0 + 2 \cdot 5 < 37 \Rightarrow \emptyset$

Hence, M_1 is critical point and it can be defined as maximizer when check SOC in it

например графически.