3. Minimize $x^2 - 2y$ subject to constraints $x^2 + y^2 \le 1$, $x \ge 0, y \ge 0$.

How will the minimal value change if the right side of the functional constraint relaxes up to 1.1?

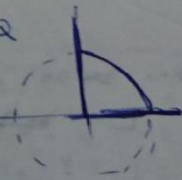
3

$$\tilde{\mathcal{L}} = x^2 - 2y + \lambda(x^2+y^2-1)$$

(Kuhn-Tucker ~~Lagrangian~~ Lagrangian is used since all constr. are inequalities and complete non-negativeness is demanded)

$$\begin{cases} \frac{\partial \tilde{\mathcal{L}}}{\partial x} \ge 0 \quad \frac{\partial \tilde{\mathcal{L}}}{\partial y} \ge 0 \\ \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} \le 0 \\ \frac{\partial \tilde{\mathcal{L}}}{\partial x} x = 0, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial y} y = 0 \\ \frac{\partial \tilde{\mathcal{L}}}{\partial \lambda} \lambda = 0 \\ x, y \ge 0, \quad \lambda \ge 0 \end{cases}$$

NDCQ



~~At most 2 constraints can be binding simultan~~

$$\mathcal{J} = \begin{pmatrix} 2x & 2y \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~At $(x,y) = (x,0)$ not any (2×2) submatrix has rank 2. But in the same time in this point constraint $(x^2+y^2 \le 1)$ which causes "problem" is not binding $\Rightarrow$ NDCQ holds.~~ $(0,y)$

$$\begin{cases} 2x + 2x\lambda \ge 0 \\ -2 + 2y\lambda \ge 0 \\ x^2+y^2-1 \le 0 \\ (2x+2x\lambda)x = 0; \quad (-2+2y\lambda)y = 0 \\ \lambda(x^2+y^2-1) = 0 \\ x, y \ge 0, \quad \lambda \ge 0 \end{cases}$$

At $\forall (x,y)^* = (0,y)$ or $(x,0)$ not any 2×2 submatrix has rank 2. NDCQ holds in all other points other than $(x,y)^*$ automatically since there ranks are full.

see cont. →

$$\begin{cases} 1+\lambda \ge 0 \\ -1 + y\lambda \ge 0 \Rightarrow y \ne 0 \Rightarrow y > 0 \\ x^2+y^2-1 \le 0 \\ (1+\lambda)x^2 = 0 \quad (*) \\ (y\lambda - 1)y = 0 \quad (**) \\ \lambda(x^2+y^2-1) = 0 \end{cases} \quad \begin{pmatrix} \lambda \ge 0 \\ x, y \ge 0 \end{pmatrix} (***)$$

continue:

in $(x,y)^*$ rank of submatrix is not full when $(x,y)^* = (0,y)$: $\begin{matrix}(x^2+y^2 \le 1) \\ (y \ge 0)\end{matrix} \begin{pmatrix} 0 & 2y \\ 0 & 1 \end{pmatrix}$, $y \ne 0$. But given $y \ge 0$ constraint, $y > 0$ holds as non-binding at this $(x,y)^* \Rightarrow$ NDCQ holds here.

$$(**) \to y\lambda - 1 = 0, \quad \lambda = \frac{1}{y} \Rightarrow x^2+y^2-1 = 0$$

$$\lambda \ne 0 \Rightarrow x = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1, \quad \lambda = 1$$

$A = (x, y, \lambda) = (0, 1, 1)$ (*** hold)

why A is ~~min to~~ minimizer, see P.S. on next page.

In $(x,y)^* = (x,0)$, $x \ne 0$: $\begin{pmatrix} 2x & 0 \\ 1 & 0 \end{pmatrix}$ has not full rank. $\leftarrow (x^2+y^2 \le 1)$, $\leftarrow x \ge 0$

But at $(x,0)$, $x \ne 0$, constraint $x \ge 0$ holds as ~~binding~~ non-binding $\Rightarrow$ NDCQ holds here as well. Hence, NDCQ holds everywhere.

From envelope theorem (for min): $\left.\frac{\partial f}{\partial b}\right|_{(x,y)^*} = -\lambda^*$ (since by construction of $\tilde{\mathcal{L}}$ for min λ is multiplied by $(-b)$)

$$\Delta f \approx \left.\frac{\partial f}{\partial b}\right|_{(x,y)^*} \Delta b = (-1)\, 0.1 = -0.1$$

$$f_{min}(A) = 0 - 2 = -2$$

Minimal value of ~~f(A)~~ f will fall by 0.1: $-2 - 0.1 = -2.1$.
This fact can also be seen from the graph on next page: when b relaxes, circle becomes of bigger radius, and point of interest, i.e. $(0,y)$, shifts upwards, $\Rightarrow$ value of $x^2 - 2y$, given x remains 0, falls.